THEORY OF THE FILTRATION OF NON-NEWTONIAN LIQUIDS THROUGH POROUS MEDIA

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The flow of a non-Newtonian liquid through soil is analyzed with an account of the channel shape.

Filtration is one of the most important processes involved in the manufacture of synthetic fibers, rubber, paints, and film materials [1-5]. The filtering materials used in this manufacture to purify solutions and melts are ceramics, metal–ceramics, quartz sand, polyvinyl chloride powder, diatomaceous earth, cotton, and fibrous and other materials [1, 5-7]. Polymer melts and solutions flow through these filtration materials through a large number of channels varying widely in cross section. Accordingly, in deriving an equation for the filtration rate of non-Newtonian liquids through porous media, one must take into account, in addition to the rheological properties, the cross-sectional shape of the channels formed in the filtration material. Below we derive a theory for filtration through a layer of grainy material.

We consider the flow of a liquid through a hypothetical soil in which the arrangement of small spheres varies between two extreme configurations: one corresponding to the closest packing and the other to the loosest packing (Fig. 1a). Here the acute angle of the rhombus, θ, varies from $\theta = 90°$ to $\theta = 60°$ (Fig. 1) [8, 9].

From Fig. 1b we see that in the closest-packing arrangement, two curvilinear triangles form on the rhombohedron faces. Accordingly, analysis of the flow of a non-Newtonian liquid through channels of triangular cross section is basic in a derivation of equations for the filtration of such liquids through a hypothetical soil. Accordingly, we will carry out a first-approximation analysis of the steady-state flow of a non-Newtonian liquid whose rheological equation is [10]

$$\eta = \eta_0 \frac{1}{1 + \alpha\tau^2}, \tag{1}$$

through a channel whose cross section is an equilateral triangle (Fig. 2). The flow curves for certain triacetate solutions which we found with a capillary viscosimeter, e.g., are described well by Eq. (1).

The equation of motion in terms of stresses can be written as

$$\frac{\partial \varepsilon_{xz}}{\partial x} + \frac{\partial \varepsilon_{yz}}{\partial y} = -\Delta p, \tag{2}$$

where τ_{xz} and τ_{yz} are the tangential stresses; Δp is the pressure drop per unit length. If we assume

$$\tau_{xz} = \frac{1}{2}\, y \frac{(x-a)}{a}\, \Delta p,$$
$$\tau_{yz} = \frac{1}{2}\left[-\frac{1}{2}(x^2 - y^2) - x\right]\Delta p, \tag{3}$$

we find that these expressions are the solutions of Eq. (2). On the other hand, we have

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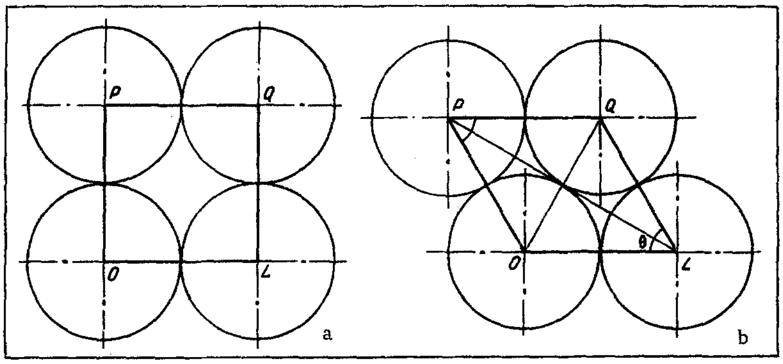
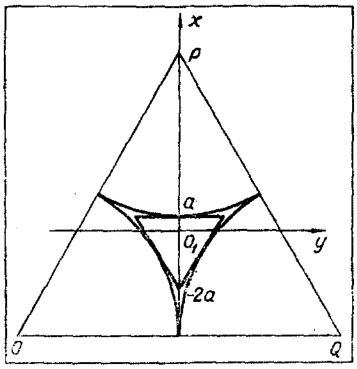


Fig. 1. Cross section of the hypothetical soil. a) Loosest packing of spheres; b) closest packing.



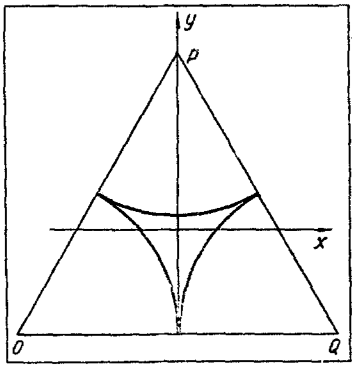


Fig. 2 Fig. 3

Fig. 2. Flow of non-Newtonian liquid through a channel whose cross section is an equilateral triangle.

Fig. 3. Flow of a non-Newtonian liquid through a channel whose cross section is a spherical triangle.

$$dV = \frac{\partial V}{\partial x}\, dx + \frac{\partial V}{\partial y}\, dy, \tag{4}$$

or

$$dV = \frac{\tau_{xz}}{\eta}\, dx + \frac{\tau_{yz}}{\eta}\, dy.$$

Using values from Eqs. (1) and (3) in Eq. (4), and integrating, we find the flow rate to be given by

$$\begin{aligned} V = {} & \frac{1}{2\eta_0}\, \Delta p \left[-\frac{1}{6a}(x^3 - 3xy^2) + \frac{2}{3}a^2 - \frac{1}{2}(x^2 + y^2) \right] \\ & + \frac{\alpha \Delta p^3}{8} \left(-\frac{x^7}{56a^3} + \frac{1}{10a^3}x^5y^2 + \frac{1}{6a}x^3y^4 - \frac{x^6}{8a^2} - \frac{y^6}{24a^2} \right. \\ & \left. - \frac{3x^5}{10a} + \frac{1}{6a^3}xy^6 - \frac{x^4}{4} - \frac{y^4}{4} - x^2y^2 + \frac{y^2x^3}{a} + \frac{3}{4a^2}y^2x^4 - \frac{7}{4a^2}x^2y^4 + \frac{3}{2a}xy^4 + \frac{97}{140}a^4 \right). \end{aligned} \tag{5}$$

We find the flow rate to be

$$Q = 2\int_{-2a}^{a}\int_{0}^{1.155a+0.5774x} V\,dx\,dy = \frac{0.78a^4 P}{\eta_0 l} + \frac{\alpha P^3 k_2 a^6}{8\eta_0 l^3}, \tag{6}$$

where P is the pressure drop. The values of $k_2 a^6$ are shown in Table 1, from which it follows that with

$$a = R\left(\frac{2}{3}\sqrt{3} - 1\right) \tag{7}$$

we have $k_2 = 3.547$.

TABLE 1. The Dependence of k_2a^6 on R

R	k_2a^6	R	k_2a^6
0,2	0,000000002	4,5	0,358124956
0,5	0,000000673	5	0,673875309
1	0,000043127	5,5	1,193811178
1,1	0,000076403	6	2,012180835
1,2	0,000128779	6,5	3,252667546
1,3	0,000208170	7	5,073968291
1,4	0,000324733	7,5	7,675841299
1,5	0,000491255	8	11,306751442
1,6	0,000723568	8,5	16,265712261
1,7	0,001041005	9	22,919997215
1,8	0,001466879	9,5	31,703058242
1,9	0,002028995	10	43,128019809
2	0,002760193	12,5	164,520341873
2,25	0,005595702	15	491,255123138
2,5	0,010529301	17,6	1238,761854079
2,75	0,018653299	20	2760,193267822
3	0,031440325	22,5	5595,702453615
3,25	0,050822930	25	10529,301879882
3,5	0,079280754	27,5	18653,300537109
3,75	0,119935332	30	31440,327880859
4	0,176652366		

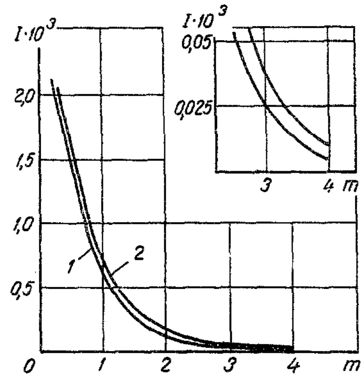


Fig. 4. The quantity I as a function of the rheological constant m: 1) $\Phi = \alpha_0 ABC$, $I_0 = 3.68 \cdot 10^{-3}$; 2) $\Phi = (\alpha_0 + \alpha_1 y) ABC$, $I_0 = 3.8 \cdot 10^{-3}$.

Equation (6) can be written as

$$Q = \frac{\omega^2 P}{34.6\eta_0 l} + \frac{\alpha\omega^3 P^3}{316.2\eta_0 l^3}, \tag{8}$$

where ω is the cross-sectional area of the triangular channel. We find the average flow velocity to be

$$V_{av} = \frac{\omega P}{34.6\eta_0 l} + \frac{\alpha\omega^2 P^3}{316.2\eta_0 l^3}. \tag{9}$$

Equation (9) is the starting point for a derivation of the equation describing the filtration rate through a hypothetical soil. According to Schlichter [8], the filtration velocity is

$$W_{av} = nV_{av}, \tag{10}$$

where n, the "clearance," is the cross-sectional area of the liquid at the narrowest part of the channel. The channel length can be evaluated by the Schlichter or Kozen' method [11].

We consider the flow of a non-Newtonian liquid through this soil, taking into account the shape of the channels which are formed in the soil in the closest-packing arrangement ($\theta = 60°$). To the best of our knowledge, an equation for the filtration rate has been derived only with an account of the kinetics of the process, without an account of other factors [12-16]. In other cases [16-18] the filtration problem has been solved only for a simplified soil structure, although rheological properties are taken into account, primarily on the basis of a power law. It follows from an analysis of the geometry of a hypothetical soil consisting of eight spheres of identical diameter, carried out by the Schlichter method [8], that two spherical triangles form in the cross section in this case (Fig. 1b). We therefore consider the flow of a non-Newtonian liquid through one such channel (Fig. 3).

For the steady-state flow of an incompressible liquid, the equation of motion in terms of stresses is

$$\frac{\partial \tau_{xz}}{\partial x'} + \frac{\partial \tau_{yz}}{\partial y} = -\Delta p. \tag{11}$$

Multiplying the left and right sides of Eq. (11) by Vdxdy, integrating, and carrying out some other manipulations, we find

$$\frac{1}{\Delta p}\iint_s \left(\tau_{yz}\frac{\partial v}{\partial y} + \tau_{xz}\frac{\partial v}{\partial x}\right) dxdy = \iint_s V dxdy, \tag{12}$$

where the right side is the liquid volume which flows through the channel per second. Since the gradient of the shear velocity and the tangential stress are related by

$$\frac{\tau_{xz}}{\eta_{ef}} = \frac{\partial v}{\partial x}, \tag{13}$$

by substituting Eq. (13) into Eq. (14) and using

$$\tau^2 = \tau_{xz}^2 + \tau_{yz}^2, \quad (14)$$

we find

$$Q = \frac{1}{\Delta p} \iint_s \varphi \tau^2 dxdy, \quad (15)$$

where φ is the fluidity of the non-Newtonian liquid; $\varphi = 1/\eta_{ef}$; η_{ef} is the effective viscosity.

The flow curves for various high-polymer solutions can be described by the rheological equation

$$\varphi = \Sigma \alpha_i \tau^{m_i}. \quad (16)$$

For the general case, taking into account rheological equation (16), find the average velocity through the channel to be

$$V = \Sigma k \iint_s \left[\left(\frac{\partial \Phi}{\partial x} \right)^2 + \left(\frac{\partial \Phi}{\partial y} \right)^2 \right]^{\frac{m}{2}+1} dxdy, \quad (17)$$

where $k = f(\alpha, m, \Delta p, \eta_{01}R)$ is a function of the soil parameters, the rheological constants, and the pressure drop. The general form of the function $\Phi(x, y)$, which satisfies boundary conditions (because of the adhesion of the liquid, it also vanishes at $V = 0$), can be written in the following manner for a cross section of a given shape:

$$\Phi = \Sigma\Sigma \alpha_{mn} x^m y^n. \quad (18)$$

For the remainder of the solution, we restrict the discussion to the first two terms in the series in Eq. (18):

$$\Phi(x, y) = (\alpha_0 + \alpha_1 y) ABC, \quad (19)$$

where A, B, C are the equations of the sides of the spherical triangle; and the coefficients α_0 and α_1 are found by the Ritz method.

Figure 4 shows, in the first and second approximations, values of the following integral as a function of the rheological constant m:

$$I = \iint_s \left[\left(\frac{\partial \Phi}{\partial x} \right)^2 + \left(\frac{\partial \Phi}{\partial y} \right)^2 \right]^{\frac{m}{2}+1} dxdy. \quad (20)$$

To calculate the filtration rate we first find the value of I from the rheological constant and Fig. 4; then, knowing the geometrical parameters of the soil and the rheological characteristics of the liquid, we can find the filtration rate through a hypothetical soil for any pressure from

$$W_f = nV. \quad (21)$$

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